$\mathcal{D}$-modules

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The long and the short of it

$\mathfrak{sl}_2$ is the tangent space of SL_2 at the identity

There is the following:

$$\begin{array}{ccc}
\mathrm{SL}_2 & \xrightarrow{\text{flag variety}} & \mathbb{P}^1 \\
\text{tangent space at identity} \Big\downarrow & & \Big\downarrow \text{global sections of tangent sheaf} \\
\mathfrak{sl}_2 & \xrightarrow{\quad \cong \quad} & \mathcal{T}(\mathbb{P}^1) \\
\text{universal enveloping algebra} \Big\downarrow & & \Big\downarrow \text{global sections of sheaf of } \textbf{differential operators} \\
\mathfrak{U}(\mathfrak{sl}_2) & \longrightarrow & \textbf{?}
\end{array}$$

Some Sheaves on X

Throughout, X is a smooth complex variety.

Definition (Sheaf of $\mathbb{C}$-linear Endomorphisms)

Let $\mathcal{M}$ be a sheaf of $\mathbb{C}$-vector spaces on X. The **sheaf of $\mathbb{C}$-linear endomorphisms** of $\mathcal{M}$, denoted by $\mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$, is the sheaf of rings

$$U \mapsto \mathrm{Hom}_{\underline{\mathbb{C}}_X}(\mathcal{M}|_U, \mathcal{M}|_U)\,,$$

where $\underline{\mathbb{C}}_X$ is the constant sheaf associated to $\mathbb{C}$.

Remark: $\mathcal{O}_X$ is a subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ via (left) multiplication.

Definition (Tangent Sheaf)

The **tangent sheaf** of X, denoted by $\mathcal{T}_X$, is the subsheaf

$$\mathcal{T}_X = \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X) = \{\theta \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \mid \theta(fg) = f\theta(g) + g\theta(f)\}\,.$$

Differential Operators and $\mathcal{D}$-modules

Definition (Sheaf of Differential Operators)

The subsheaf of $\mathbb{C}$-algebras generated by $\mathcal{O}_X$ and $\mathcal{T}_X$ in $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ is the **sheaf of differential operators** on X, denoted by $\mathcal{D}_X$.

Definition ($\mathcal{D}$-module)

Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{M}$ is a **(left) $\mathcal{D}$-module** if $\mathcal{M}(U)$ is endowed with the structure of a (left) $\mathcal{D}_X(U)$-module for every open subset $U \subseteq X$ and these structures are compatible with restrictions.

Remark: A (left) $\mathcal{D}$-module $\mathcal{M}$ may, equivalently, be given as a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ equipped with a $\underline{\mathbb{C}}_X$-linear morphism of sheaves $\nabla \colon \mathcal{T}_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$ satisfying, in particular, a Leibniz-type rule.

Example: $X = \mathbb{A}^n$

Sheaves on affine n-space $\mathbb{A}^n$ are determined by their global sections

Example: $X = \mathbb{A}^n$

Since X is affine, $\mathcal{D}(\mathbb{A}^n) = \mathcal{D}_X(X)$ is a $\mathbb{C}$-subalgebra of

$$\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)(X) = \mathrm{End}_{\mathbb{C}}(\mathcal{O}_X(X)) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_n])\,.$$

This subalgebra is spanned by the elements of

$$\mathcal{O}_X(X) = \mathbb{C}[x_1, \ldots, x_n] \quad \text{and} \quad \mathcal{T}_X(X) = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \ldots, x_n]\partial_{x_i}$$

where $f \in \mathbb{C}[x_1, \ldots, x_n]$ acts by (left) multiplication and ∂_{x_i} acts by taking the partial derivative with respect to x_i.

Since endomorphism rings are generally noncommutative, we have to compute the commutator relations for the $2n$ generators $x_1, \ldots, x_n$ and $\partial_{x_1}, \ldots, \partial_{x_n}$. These are

$$[x_i, x_j](f) = x_i(x_j f) - x_j(x_i f) = 0$$
$$[\partial_{x_i}, \partial_{x_j}](f) = \partial_{x_i}(\partial_{x_j} f) - \partial_{x_j}(\partial_{x_i} f) = 0$$
$$[\partial_{x_i}, x_j](f) = \partial_{x_i}(x_j f) - x_j(\partial_{x_i} f) = \delta_{ij} f + x_j(\partial_{x_i} f) - x_j(\partial_{x_i} f) = \delta_{ij} f$$

Example: $X = \mathbb{A}^n$

Thus, we can identify $\mathcal{D}(\mathbb{A}^n)$ with the ***n*-th Weyl algebra $A_n(\mathbb{C})$**, which is defined as

$$A_n(\mathbb{C}) = \frac{\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle}{\langle [x_i, x_j], [y_i, y_j], [x_i, y_j] - \delta_{ij}\rangle}.$$

Consequently, (left) $\mathcal{D}$-modules correspond to (left) modules over the (noncommutative) Weyl algebras.

Example: $X = \mathbb{P}^1$

Sheaves on projective n-space $\mathbb{P}^n$ are determined by their restrictions to the standard affine cover by affine n-spaces $\mathbb{A}^n$ together with glueing conditions for their intersections

Example: $X = \mathbb{P}^1$

Let $U_0 = \{[x_0 : x_1] \mid x_0 \neq 0\}$ and $U_1 = \{[x_0 : x_1] \mid x_1 \neq 0\}$ be the standard affine cover of X. Then $U_0 \cong U_1 \cong \mathbb{A}^1$ and $U_0 \cap U_1 \cong D(0) = \mathbb{A}^1 \setminus \{0\}$.

Choosing coordinates, we have $\mathcal{O}_X(U_0) = \mathbb{C}[t]$, $\mathcal{O}_X(U_1) = \mathbb{C}[s]$ and $\mathcal{O}_X(U_0 \cap U_1) = \mathbb{C}[t, t^{-1}]$ with $s = t^{-1}$. Then $\mathcal{D}_X(U_0)$ is generated by t and ∂_t and $\mathcal{D}_X(U_1)$ is generated by s and ∂_s. On the intersection, we find that

$$\partial_t(s) = \partial_t(t^{-1}) = -t^{-2}\partial_s(s) \implies \partial_s = -t^2\partial_t$$

as the defining relation. Here we use that $\mathcal{D}_X(U_0 \cap U_1)$ is generated by t, t^{-1} and ∂_t, being the restriction of $\mathcal{D}_X(U_0)$ to a distinguished open. Consequently, a $\mathcal{D}$-module on X is given by a pair of $A_1(\mathbb{C})$-modules, whose localisations at t and $s = t^{-1}$, respectively, are isomorphic.

Remark: In contrast to general $\mathcal{O}_X$-modules on $\mathbb{P}^1$, it turns out that $\mathcal{D}$-modules on $\mathbb{P}^1$ are completely determined by their global sections.

$\mathcal{T}(\mathbb{P}^1)$ as Lie algebra

$\mathcal{T}(\mathbb{P}^1)$ together with the commutator of derivations is a Lie algebra, which is isomorphic to $\mathfrak{sl}_2$

$\mathcal{T}(\mathbb{P}^1)$ as Lie algebra

We consider the standard affine cover as before. Since $\mathcal{T}_X$ is a sheaf, we can compute $\mathcal{T}(\mathbb{P}^1) = \mathcal{T}_X(X)$ from the corresponding exact sequence:

$$0 \longrightarrow \mathcal{T}(\mathbb{P}^1) \longrightarrow \mathcal{T}_X(U_0) \times \mathcal{T}_X(U_1) \longrightarrow \mathcal{T}_X(U_0 \cap U_1)\,.$$

In the given coordinates, we know that $\mathcal{T}_X(U_0) = \mathbb{C}[t]\partial_t$, $\mathcal{T}_X(U_1) = \mathbb{C}[s]\partial_s$, $\mathcal{T}_X(U_0 \cap U_1) = \mathbb{C}[t, t^{-1}]\partial_t$ with $s = t^{-1}$ and $\partial_s = -t^2\partial_t$.

A pair $(f(t)\partial_t, g(s)\partial_s)$ defines an element of $\mathcal{T}(\mathbb{P}^1)$ iff

$$f(t)\partial_t - g(t^{-1})(-t^2\partial_t) = 0 \iff$$
$$a_0\partial_t + a_1 t\partial_t + a_2 t^2\partial_t + \cdots = -(b_0 t^2\partial_t + b_1 t\partial_t + b_2\partial_t + \cdots)\,.$$

This is only possible if $\deg f(t), \deg g(s) \leq 2$. Thus, $\mathcal{T}(\mathbb{P}^1)$ can be identified with the subspace of $\mathcal{T}_X(U_0) = \mathbb{C}[t]\partial_t$ generated by ∂_t, $t\partial_t$ and $t^2\partial_t$.

$\mathcal{T}(\mathbb{P}^1)$ as Lie algebra

Denote $e = -\partial_t$, $f = t^2\partial_t$ and $h = -2t\partial_t$. Then

$$[h, e] = [-2t\partial_t, -\partial_t] = 2t[\partial_t, \partial_t] + [2t, \partial_t]\partial_t = -2\partial_t = 2e\,.$$

Similarly, $[h, f] = -2f$ as well as $[e, f] = h$. This defines an isomorphism of $\mathfrak{sl}_2$ and $\mathcal{T}(\mathbb{P}^1)$ as Lie algebras. Since $\mathcal{O}(\mathbb{P}^1) = \mathbb{C}$, $\mathcal{T}(\mathbb{P}^1)$ generates $\mathcal{D}(\mathbb{P}^1)$ as $\mathbb{C}$-algebra and there is an induced morphism

$$\mathfrak{U}(\mathfrak{sl}_2) \to \mathcal{D}(\mathbb{P}^1)$$

from the universal enveloping algebra of $\mathfrak{sl}_2$ to $\mathcal{D}(\mathbb{P}^1)$. We know that the centre of $\mathfrak{U}(\mathfrak{sl}_2)$ is generated by the Casimir element

$$\Omega = 2\left(\tfrac{1}{2}h^2 + ef + fe\right) = h^2 + 2(ef - fe) + 4fe = h^2 + 2h + 4fe\,.$$

One checks that

$$\begin{aligned}(-2t\partial_t)^2 + 2(-2t\partial_t) + 4(t^2\partial_t)(-\partial_t) &= 4t([\partial_t, t] + t\partial_t)\partial_t - 4t\partial_t - 4t^2\partial_t^2 \\ &= 4t(1 + t\partial_t)\partial_t - 4t(1 + t\partial_t)\partial_t = 0\,.\end{aligned}$$

Part of the assertion of Beilinson–Bernstein's localisation theorem is that this induces an isomorphism

$$\mathfrak{U}(\mathfrak{sl}_2)/\mathrm{Z}(\mathfrak{U}(\mathfrak{sl}_2)) \cdot \mathfrak{U}(\mathfrak{sl}_2) \xrightarrow{\sim} \mathcal{D}(\mathbb{P}^1)\,.$$

A Final Remark

There is a natural (algebraic) action of SL_2 on $\mathbb{P}^1$ via linear fractional transformations, which yields the map $\mathfrak{sl}_2 \to \mathcal{T}(\mathbb{P}^1)$ by (algebraic) differentiation

Sketch: Intrinsic Construction of $\mathfrak{U}(\mathfrak{sl}_2) \to \mathcal{D}(\mathbb{P}^1)$

The map $\mathfrak{U}(\mathfrak{sl}_2) \to \mathcal{D}(\mathbb{P}^1)$ admits an intrinsic description, using:

(1) $\mathbb{P}^1$ is the *flag variety* of SL_2, admitting a natural SL_2-action;

(2) $\mathfrak{sl}_2$ is the tangent space of SL_2 at the identity.

Sketch: Intrinsic Construction of $\mathfrak{U}(\mathfrak{sl}_2) \to \mathcal{D}(\mathbb{P}^1)$

We want to construct a map $\mathfrak{U}(\mathfrak{sl}_2) \to \mathcal{D}(\mathbb{P}^1)$. We know that $\mathcal{D}(\mathbb{P}^1)$ is generated, as $\mathbb{C}$-algebra, by $\mathcal{T}(\mathbb{P}^1)$. So it suffices to associate to any element of $\mathfrak{sl}_2$ a derivation of sheaves $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}$, which is determined by its restriction to the standard affine cover.

On the affine chart U_0, SL_2 acts by linear fractional transformations on $t \in U_0$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot t = \frac{at+b}{ct+d} .$$

This induces an action on $f \in \mathcal{O}_{\mathbb{P}^1}(U_0)$:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(t) = f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot t\right) = f\left(\frac{dt-b}{a-ct}\right) .$$

This naturally leads to a realisation of the elements of $\mathfrak{sl}_2$ as derivations of $\mathcal{O}_{\mathbb{P}^1}(U_0)$.

Sketch: Intrinsic Construction of $\mathfrak{U}(\mathfrak{sl}_2) \to \mathcal{D}(\mathbb{P}^1)$

Let $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$. Since $\mathfrak{sl}_2$ is the tangent space of SL_2 at the identity, there is an isomorphism

$$\mathfrak{sl}_2 \to \mathrm{SL}_2(\mathbb{C}[\epsilon])_I, \qquad M \mapsto I + \epsilon M$$

which identifies e with the invertible matrix $E = \left(\begin{smallmatrix} 1 & \epsilon \\ 0 & 1 \end{smallmatrix}\right)$. Here $\mathbb{C}[\epsilon]$ denotes the dual numbers, hence $\epsilon^2 = 0$. From this we obtain

$$(E \cdot f)(t) = f(E^{-1} \cdot t) = f(t - \epsilon) = f(t) + (-f'(t))\epsilon\,.$$

Thus, the differential of E, which by construction describes the induced action of e, is given by the derivation $-\partial_t$, as expected. A similar computation associates to f the derivation $t^2\partial_t$ and to h the derivation $-2t\partial_t$, respectively.